The Spatial Capacity of a Slotted ALOHA Multihop Packet Radio Network with Capture

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Abstract—In this paper we determine throughput equations for a packet radio network where terminals are randomly distributed on the plane, are able to capture transmitted signals, and use slotted ALOHA to access the channel. We find that the throughput of the network is a strictly increasing function of the receiver's ability to capture signals, and depends on the transmission range of the terminals and their probability of transmitting packets. Under ideal circumstances, we show the expected fraction of terminals in the network that are engaged in successful traffic in any slot does not exceed 21 percent.

I. Introduction

THE proliferation of computers within the last decade has created the need to interconnect computing resources with efficient and economical communications. Packet radio broadcast techniques have been proposed as a method to implement such computer networks and are an attractive alternative to conventional land-based line networks because radio networks are not dependent on fixed topologies, can be connected to numerous devices, and can be implemented with inexpensive radio transceivers. Many novel uses for future ratio networks can be found in [1]. The first packet radio network, the ALOHA system [2], [3], demonstrated the feasibility of this approach as did the PRNET of the Defense Advanced Research Projects Agency [4]. Nodes in radio networks, known as terminals, are geographically separated and can communicate only by use of the broadcast channel. It is thus important to develop techniques that make efficient use of channel bandwidth. There has been extensive research in creating efficient protocols for networks in which all terminals are assumed to be within line-of-sight of each other [5]. In such *one-hop* networks, all terminals share common information about the status of the channel. The channel is said to be *idle* if no terminals transmit, *successful* if exactly one terminal transmits, and to have a *collision* if two or more transmit simultaneously. In networks where packets must be relayed over several hops before reaching their final destination, the status of the channel can be known immediately only within the hearing distance of a terminal. Since a transmitted packet is received by only a subset of the nodes in the network, there is the possibility that another terminal in a different part of the network may also be successfully transmitting a packet during the same time since it is not disturbed by the first terminal's transmission. This important phenomenon is called *spatial-reuse* of the channel and was studied in [6]. The local nature of the channel state information, however, causes difficulty in coordinating the transmissions of terminals. In single-hop environments, terminals share common information about the status of the channel. This information can be used to develop efficient single-hop channel access protocols; however, such protocols cannot always be readily adapted to multihop environments.

Another example in the one-hop environment is the CSMA protocol [7]-[9]. In this protocol, a terminal first senses the channel to determine if the channel is idle. If so, the terminal transmits its packet immediately. The probability of a collision on the channel is equal to the probability that two or more terminals simultaneously sensed the channel idle and transmitted their packets. For networks that are not separated by vast distances, the propagation delay between terminals is small enough to make this event occur infrequently. In the multihop environment, however, hearing the channel idle provides information only about the transmitter's local environment and does not guarantee that the receiver's environment is also idle. Thus the probability of incurring a collision is no longer only a function of the propagation delay and packets will often collide [10].

This implies that using protocols developed for a single-hop environments will not always perform well in multihop networks. The performance of the slotted ALOHA protocol in a multihop environment has been studied in [11] and in [12] the authors calculated the transmission radius that maximized throughput for a random planar network. In this paper we generalize their work to environments where radio receivers have the ability to *capture* signals. A receiver equipped with capture can, under certain circumstances, successfully decode one of several simultaneous signals on the channel. Capture models for single-hop configurations have been studied [13], [14], and demonstrate increased performance over noncapture environments. In this paper we show that for a multihop random network increasing the receiver's ability to capture signals will always increase the throughput of the network.

II. The Model

Throughout the paper we will make the following assumptions about the network:

1) Topology: We assume that packet radios are distributed according to a Poisson point process on the plane with a mean density of λ packet radio units (also called terminals) per unit area. We are interested in finding the throughput for an area containing a large number n of packet radios and will ignore edge effects. This topology represents an instantaneous snapshot of a mobile packet radio network.

2) Stations: We assume that each packet radio transmits with fixed power, and that all n stations (i.e., terminals) in the network transmit with the same power on the same frequency band. Receivers are assumed to be able to receive a signal from another station if that station is within a radius R of the transmitter, and under certain circumstances, can successfully *capture* one of several simultaneous transmissions within its hearing range. Let us describe the phenomenon of capture by considering a receiver a which is within range of two transmitters t_1 and t_2 and assuming that t_1 has a packet destined for a. Let P_1 and P_2 be the powers of the signals received by a, and r_1 and r_2 be the distances between a and the two transmitters. Whenever both t_1 and t_2 transmit

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their packets simultaneously, their signals interfere with each other. In the absence of capture, station t_1 will not be received correctly. With capture however, station a can successfully receive t_1's transmission if $P_1/P_2 > \beta^{-1} (0 \leqslant \beta \leqslant 1)$, where β is called the *capture-ratio*. Assuming omnidirectional antennas on the plane and equal transmitting power for all stations, this ratio of powers can be converted into a ratio of distances since the power of a received signal decreases as the inverse square of the distance. Thus, using this distance measure, a will capture t_1 if $r_2/r_1 > \beta^{-1/2}$. From Fig. 1 we see that t_1 will be successful if t_2 lies outside the circle of radius $r\beta^{-1/2}$, called the *capture radius*. Observe that $\beta = 0$ implies that simultaneous transmission will always cause a collision (*noncapture*), and $\beta = 1$ implies t_1 will be received if it is simply closer to a than t_2 (*perfect-capture*). Well-designed FM receivers have a capture ratio approximately equal to 0.7 [13]. Although the appearance of the exponent $(-1/2)$ of β in the above equation appears awkward, this selection simplifies later equations. If a receiver captures the stronger of two signals in our model, the weaker station's signal is essentially considered to be noise. Thus the capture parameter of the receiver is a function of the minimum signal-to-noise ratio that is necessary for correct detection of signals on the channel. We will assume that packets contain a checksum which is utilized to detect collisions.

3) Channel Access Method: We assume that the time axis is slotted and will analyze the network under the heavy traffic assumption. In particular, we will assume that each terminal has an infinite queue of packets, the first of which is transmitted with probability p in each slot. This assumption corresponds to running the network at channel capacity at which point, for most queueing systems, the queue lengths of the nodes of the network grow without bound. We should note that the stability of slotted-ALOHA in the multihop environment at input rates less than channel capacity is not at issue in this work.

4) Traffic Matrix: Since nodes are Poisson distributed on the plane with mean density λ, and since any station can send and receive packets within a radius R, every station has on the average $N = \lambda\pi R^2$ neighbors (terminals within its hearing and transmitting range). We assume the global traffic matrix for all the n nodes in the network is uniform, and thus the probability of sending to any particular node in the network is $1/n$.

5) Routing: We choose to study the case where packets destined toward a particular node F in the network are routed with equal probability towards one immediate neighboring node that lies in the general direction of F. For example, in Fig. 2 there are k terminals lying in the direction from transmitter t to final destination F. Transmitter t will pick one terminal from its k neighbors with probability $1/k$.

These assumptions are similar to those presented in [12] except for the addition of capture and the nature of the routing algorithm. Let us justify the random routing assumption by comparing it to an optimal routing model. In [12], packets are assumed to be relayed to a neighboring terminal that is closest (furthest along the path) to that packet's final destination. This optimal routing policy is not realizable for a mobile packet radio network since it requires the exact location of all terminals (which are assumed to be moving) as well as that of the final destination, and thus provides an upper bound for network performance. This upper bound can be easily calculated in networks without capture because the probability of being successfully received is independent of the distance between the transmitter and receiver. In the capture environment however, nodes closer to the transmitter have a greater probability of receiving a transmitted signal than those further away. This nonuniformity makes the calculation of the distance covered in one transmission for an optimal routing policy difficult. To be specific, suppose a transmitter t has a message to send to a particular final node F (Fig. 2). Suppose t has k neighbors lying in the half circle of his transmission radius, toward F, and that their distances from t are $(r_1, r_2, \cdots, r_k)$. Let $(z_1, z_2, \cdots, z_k)$ be the vector of projected distances toward F; hence, if t transmits to node i at r_i, the progress toward F will be z_i. Let $P(r_i)$ be the probability that node i successfully receives t's transmission. A locally optimal routing algorithm for this system is defined as one that sends all packets toward F to the node j that lies closest to F. If t sends to node a_i, then the expected forward progress for this transmission is equal to $z_iP(r_i)$, and thus an optimal routing algorithm would send packets to the node with the maximum value of $z_iP(r_i)$. To determine this value, one must calculate the joint probability for $(r_1, r_2, \cdots, r_k)$ and $(z_1, z_2, \cdots, z_k)$ for all k, to determine the density for the maximum projected distance. This would then be unconditioned on k to determine the density for the maximal forward progress. In the noncapture environment $P(r_i) = P(r_j)$ for all i and j, and thus maximizing $z_iP(r_i)$ implies picking the maximum z_i as in [12]. Over all sets of k nodes, the probability that the maximum projected distance is equal to a certain value, say z, is seen to be the probability that there are no terminals in the half circle from t that are closer to F (the shaded region A in Fig. 3). Since terminals are Poisson distributed on the plane, this probability equals $e^{-\lambda A}$. In the capture environment however, $P(r_i) = P(r_j)$ only if $r_i = r_j$, and the above calculation is no longer valid. The optimal routing algorithm now will no longer always pick the node with the maximum z_i because the product $z_jP(r_i)$ may not be maximal. To avoid these computational complexities and in endeavoring to create a practical bound for packet radio networks, we have chosen to assume a random routing policy (assumption 5) above). Random policies have been proposed for networks of this kind [15] and our calculations will be a lower bound on the performance for algorithms that always send packets in the direction of their final destination.

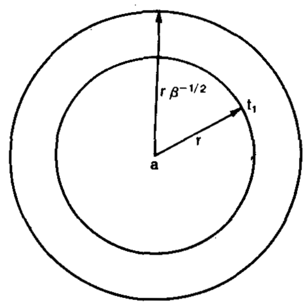

Fig. 1. Defining the *capture ratio*.

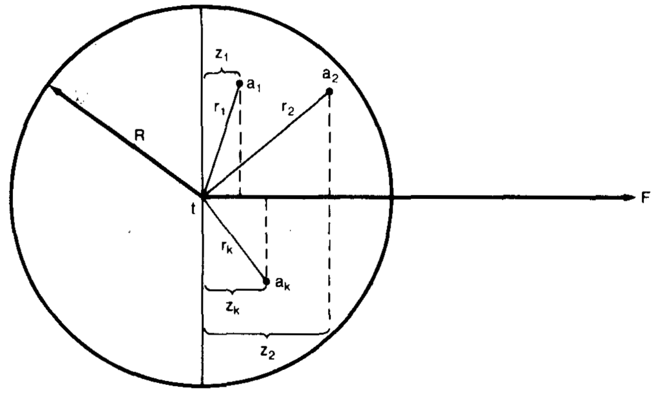

Fig. 2. An optimal routing policy.

With this random routing assumption we will analyze two models that differ in the way capture is defined in relation

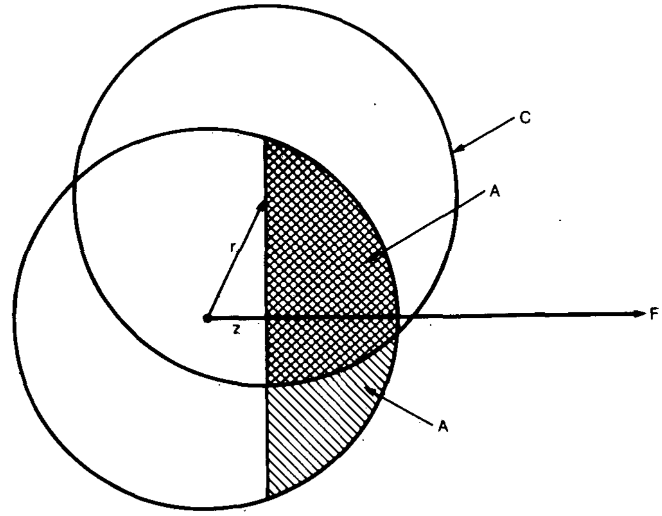

Fig. 3. Probability of successful reception.

to the maximum transmission distance R. Continuing with the scenario of 2), it is certainly true that in the presence of multiple simultaneous transmissions receiver a can only successfully receive packets from its closest transmitting neighbor. Thus we can say that, in the presence of simultaneous transmission, a receives from t_1 if t_1 is a's nearest transmitter, and there are no other transmitters within the capture radius. Let us therefore suppose in our scenario that t_i is a's closest transmitter and is located a distance r away. Our two models differ in the way they define the capture radius in relation to the maximum transmitting distance R.

In the first case we assume that R is a hard boundary. Suppose then that $r\beta^{-1/2} > R$, or in words, that the capture radius is greater than a's maximum hearing distance. A transmitter located further than R, say at r' with $R < r' < r\beta^{-1/2}$ has no effect on a's reception since its signal is too weak to be received. In Model 1 we assume such a boundary and define the capture radius to be equal to the minimum of $r\beta^{-1/2}$ and R. The area that must contain no other transmitters for a to successfully receive t_1's transmission, the *clean area*, for Model 1 is the annulus of inner radius equal to r and outer radius equal to the minimum of $r\beta^{-1/2}$ and R.

This definition for the capture radius for Model 1, however, gives weak signals coming from a transmitter located at a distance slightly less than R, say at $R - \epsilon$, a greater probability of being successfully received than a transmitter with a smaller value of r since the clean area, the annulus of inner radius $R - \epsilon$ and width ϵ, is infinitesimal. In practice, however, the boundary is not hard and if there is a transmitter at a distance slightly greater than R, say at $R + \epsilon$, it would disrupt reception since the ratio of the powers of the two transmitters would be close to 1 even though the second transmitter's signal is very weak. Model 2 attempts to account for this discrepancy by defining the clean area to be $r\beta^{-1/2}$ regardless of the relationship between $r\beta^{-1/2}$ and R.

We observe since the clean area for perfect capture ($\beta = 1$) is identical in both models, we expect our equations to be the same for this case. In fact, both models are very similar for $\beta > 0.7$. We must comment that both models make two simplifying assumptions about the capture phenomenon. In actual pratice, one particular transmitter, say t_1, will be captured by a certain receiver if the ratio of its received power, to the sum of the received powers of *all* other signals simultaneously heard by the receiver, is greater than the specified capture ratio. Letting P_{t_i} be the receiver power for the ith transmitter k be the number of transmitters the receiver hears, and assuming that the powers are sorted into decreasing order ($P_{t_1} > P_{t_2} > \cdots > P_{t_k}$), we have that t_1 is captured if $P_{t_1}/\Sigma_{i=2}^{k} P_{t_i} > \beta$. In our models we approximate the sum of all the powers of terminals t_2 through t_k by the power of the next strongest signal P_{t_2}. This assumption, however, is not critical for optimized networks, as we will later see, since the probability that there are more than two transmitters within range of a given receiver is very small.

The second simplifying assumption we make is that capture is a deterministic phenomenon such that if the ratio of the received powers is greater than β then the signal is captured with probability 1. In actuality however, capture is probabilistic and has a density that is a function of the ratio of the received powers and of the capture parameter. The results of our deterministic model can be applied to this more realistic model, however, without too much error, by using a value of β so that if $P_{t_1}/\Sigma_{i=2}^{k} P_{t_i} > \beta$ then the actual probability of being captured is greater than some specified confidence probability (say 0.95).

III. Analysis of Model 1

A. Expected Number of Successful Receptions

We first calculate the probability of successful reception for a randomly selected terminal in the network. Let us assume that terminal a captures the transmission of its closest transmitting neightbor t. Conditioned on this, a will successfully receive t's packet if the packet was addressed to a and if a did not transmit in the current slot. This occurs with probability

$$P[S \mid \text{no interference}] = \frac{(1-p)(1-e^{-N/2})}{N}$$

where S is the event that a randomly selected terminal successfully receives a packet in a randomly selected slot, p is the probability of transmitting in the next slot, and N is the average number of neighboring nodes. We can see this by first defining the following events.

T: The event that t sends to a.

D: The event that t sends in the half circle that contains a.

$N(i)$: The probability that there are i other terminals besides a in the half circle of radius R from t.

In Fig. 2, for example, there are k terminals in the half circle from t toward F, and transmitter t is sending in the direction of all the labeled terminals in the figure.

We know that

$$P[T] = P[T \mid D]P[D] + P[T \mid D^c]P[D^c]$$

but $P[T|D^c] = 0$ since we do not allow packets to go away from their destination. Since t's destination is uniformly distributed over the plane, the probability that a randomly chosen terminal a within a radius R of t is in the direction of F, is equal to the probability that a lies in the half circle of radius R directed from t to F. Since a is equally likely to be in either half we have $P[D] = 1/2$. Calculating the remaining term, $P[T|D]$, we have

$$P[T \mid D] = \sum_{i=0}^{\infty} P[T \mid D, N(i)]P[N(i) \mid D].$$

Each of these terms is known, for $P[N(i)|D] = P[N(i)]$ is Poisson with parameter $\lambda\pi R^2/2$ and $P[T|D, N(i)] = 1/(i + 1)$ since if there are i other terminals besides a, making a total of $i + 1$ terminals, t will select one of them with equal probability. Recalling that the average number of neighbors is

$N = \lambda\pi R^2$, we have

$$P[T \mid D] = \sum_{i=0}^{\infty} \frac{1}{i+1} \frac{e^{-N/2}(N/2)^i}{i!} = \frac{2}{N}(1 - e^{-N/2}).$$

Combining with the previous calculations we obtain

$$P[T] = \tfrac{1}{2}P[T \mid D] = \frac{(1 - e^{-N/2})}{N}.$$

Knowing that the packet was addressed to a in the absence of interference we know that a will successfully receive t's packet if a does not transmit, thus giving the $(1 - p)$ term and establishing the above expression.

We must now calculate the probability that there is no interfering traffic. We do this by first conditioning on the distance between a and t to be $r (r \leqslant R)$, and then analyzing two cases.

Case 1) $\beta^{-1/2}r < R$ [Fig. 4(a)]: In this case a will receive the packet if there are no other transmitters in $(r, r\beta^{-1/2})$, the *clean area*. This area is equal to $\pi(\beta^{-1/2}r)^2 - \pi r^2 = \pi r^2(1/\beta - 1)$ which contains no transmitters with probability $e^{-\lambda p \pi r^2 (1/\beta - 1)}$.

Case 2) $\beta^{-1/2}r > R$ [Fig. 4(b)]: The area that must now be clean is see to be $(\pi R^2 - \pi r^2)$ which occurs with probability $e^{-\lambda p \pi (R^2 - r^2)}$.

The density for the distance between a randomly selected terminal and its closest transmitting neighbor can be easily calculated. Letting X be the random variable for this distance, and knowing that busy terminals are Poisson distributed on the plane with parameter λp we have

$$P[X \leqslant r] = 1 - P\,[\text{no busy terminal in } (0, r)] = 1 - e^{-\lambda\pi p r^2}.$$

Hence, letting $f(r)$ be the density for X

$$f(r) = \frac{dP[X \leqslant r]}{dr} = 2\pi\lambda p r e^{-\lambda p \pi r^2}.$$

Using this in the above, after unconditioning we obtain

$$P[S] = \frac{(1-p)(1-e^{-N/2})}{N} \cdot \left[\int_0^{R\beta^{1/2}} e^{-\lambda p \pi r^2 (1/\beta - 1)} 2\pi\lambda p r e^{-\lambda p \pi r^2}\, dr + \int_{R\beta^{1/2}}^{R} e^{-\lambda p \pi (R^2 - r^2)} 2\pi\lambda p r e^{-\lambda p \pi r^2}\, dr \right].$$

Performing the integration we have

$$P[S] = \frac{(1-p)(1-e^{-N/2})}{N} [\beta(1 - e^{-Np}) + (1-\beta)Npe^{-Np}].$$

For further equations let $Y \triangleq \beta(1 - e^{-Np}) + (1 - \beta)Npe^{-Np}$.

We will discuss two special cases to see that they are intuitively plausible.

Case 1) $\beta = 0$: The noncapture case for which

$$P[S] = (1-p)(1 - e^{-N/2})pe^{-Np}.$$

Intuitive explanation—For a to receive it must not transmit and this occurs with probability $(1 - p)$. Receiver a also cannot

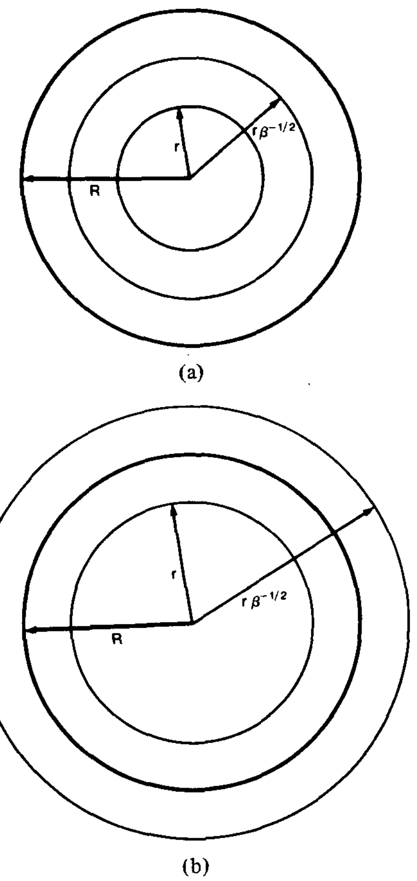

Fig. 4. (a) Case 1. Probability of no interfering traffic for $\beta^{-1/2}r < R$. (b) Case 2. Probability of no interfering traffic for $\beta^{-1/2}r \geq R$.

be isolated from other nodes in the network. In particular, one half circle of radius R must contain at least one other terminal and this occurs with probability $(1 - e^{-N/2})$. Out of the neighbors in this region only 1 can transmit (occurring with probability Npe^{-Np}), and on the average that transmitter is surrounded by N neighbors and thus transmits to receiver a with probability $1/N$. Combining all the above probabilities we obtain the above expression.

Case 2) $\beta = 1$: The perfect capture case where

$$P[S] = \frac{(1-p)(1-e^{-N/2})}{N}(1 - e^{-Np}).$$

Intuitive explanation—Again, a must be silent which occurs with probability $(1 - p)$, and must not be isolated from nodes in one half circle of radius R which occurs with probability $(1 - e^{-N/2})$. There must be at least one transmitting station in its neighborhood (occurring with probability $1 - e^{-Np}$) and since only a's nearest neighbor can be successfully received by a, the probability it transmits a packet to a on the average is $1/N$.

To calculate the expected number of successes in the network (denoted by $\#S$) we merely have to multiply the previous probability by the number of nodes n in the network to obtain

$$E[\#S] = nP[S] = \frac{n}{N}(1-p)(1 - e^{-N/2})Y. \qquad (1)$$

We can check this equation against the well-known single-hop slotted ALOHA results for the infinite population model with Poisson traffic statistics. We do this by setting, in (1),

$n = N$ and $\beta = 0$, and then performing two limit operations. By letting $p \to 0$ and $n \to \infty$ in such a way as to preserve the product $G = np$ to be a constant, we obtain Poisson traffic characteristics with parameter G. We then obtain

$$\lim_{\substack{n=N\to\infty \\ p\to 0 \\ G=np}} \frac{n}{N}(1-p)(1-e^{-N/2})Npe^{-Np} = Ge^{-G}.$$

This reaches its maximum at $G = 1$ giving the familiar maximum throughput for the slotted ALOHA channel [13] of $1/e$.

It can be seen that the function $E[\#S]$ is increasing in β, the capture-parameter, by writing $E[\#S]$ as a linear function of β.

$$E[\#S] = H(\beta) = K(\beta(1-e^{-Np}(1-Np)) + Npe^{-Np})$$

where $K = (n/N)(1-p)(1-e^{-N/2})$. This describes a straight line with slope $m = 1 - e^{-Np}(1-Np)$. But this slope is always positive since $m < 0$ implies $e^{NP} < 1 - Np$ which is false. We conclude that increasing the receiver's ability to capture signals increases the expected number of successes in the network.

We now seek to determine the maximum number of expected successes in the network. Certainly this number must be less than $n/2$ since every successful receiver is associated with exactly one successful transmitter. It is easy to verify this analytically. Numerically calculating the maximum of $E[\#S]$ for various values of β demonstrates (see Table I) that the maximum expected number of terminals in the network that could engage in successful communication at any given slot is about 21 percent for perfect capture and about 14 percent for the noncapture environment (observe that these figures are double of those of Table I since every successful receiver is associated with a successful transmitter). We note here that the values of N and p that maximize the probability of success do not also maximize the throughput of the network. This is a result of the dependency of the probability of successful transmission and the maximum transmission range R as manifest in the equation $N = \lambda\pi R^2$, and will be discussed in greater length in the next section. Observe that the results of Table I would only be applicable to networks in which all packets went exactly one hop to reach their final destination.

TABLE I
OPTIMAL N AND p FOR A SINGLE HOP IN THE FIRST MODEL FOR A GIVEN β

β	N	p	$P[E_s]$
0.0	1.9880	.29377	.07280
0.1	2.0594	.29974	.07557
0.2	2.1365	.30594	.07846
0.3	2.2195	.31239	.08154
0.4	2.3085	.31963	.08484
0.5	2.3036	.32585	.08835
0.6	2.5044	.33276	.09210
0.7	2.6102	.33970	.09609
0.8	2.7201	.34659	.10030
0.9	2.8326	.35331	.10477
1.0	2.9462	.35977	.10946

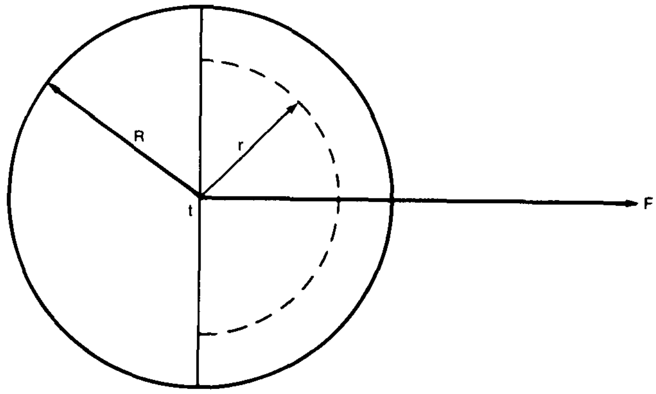

Fig. 5. Calculating the density between the transmitter and the receiver.

B. Expected Forward Progress

We are now in a position to derive the density for the distance between transmitter and receiver for a successful transmission. If we define X to be the random variable associated with the distance between a transmitter and its intended receiver (r in Fig. 5) we can write

$$P[r \leqslant X \leqslant r + dr] = \frac{2r}{R^2}\,dr. \tag{3}$$

Thus, we have that

$$P[r \leqslant X \leqslant r + dr \mid S] = \frac{P[r \leqslant X \leqslant r + dr, S]}{P[S]}$$

$$= \frac{p(1-p)(1-e^{-N/2})}{P[S]} \frac{2r}{R^2} \cdot e^{-\lambda\pi p[\min(r\beta^{-1/2}R)]^2}\,dr \tag{4}$$

or defining $q(r)$ to be the density for X and $1/s = (Np)/Y$ we may rewrite (4) to obtain

$$g(r) = \begin{cases} \dfrac{1}{s}\dfrac{2r}{R^2} e^{-\lambda p\pi r^2/\beta}; & 0 \leqslant r \leqslant R\beta^{1/2} \\ \dfrac{1}{s}\dfrac{2r}{R^2} e^{-\lambda p\pi R^2}; & R\beta^{1/2} \leqslant r \leqslant R. \end{cases} \tag{5}$$

It can be easily verified that this integrates to 1 and thus is a proper density.

Suppose as shown in Fig. 6, transmitter t is sending a packet to final destination F through intermediate node a. We wish to calculate the progress of the packet towards its final destination. To simplify the calculation we assume that forward progress will be the same for any node on the line perpendicular to the direction of the destination, line L in the figure. This assumption is reasonable if the distance D is much greater than R. Because terminals are randomly distributed on the plane, for a given distance r and a given destination F, the angle θ will be uniformly distributed over $(-\pi/2, \pi/2)$. Define Z to be the random variable denoting the forward distance. We see that for a given r the probability that Z is less than some value z is the same as the probability of $|\theta|$ being larger than $\cos^{-1}(z/r)$, or letting $F(z)$ be the distribution of Z we have

$$F(z \mid r) = \begin{cases} 1; & r < z \\ 1 - \dfrac{2\cos^{-1}(z/r)}{\pi}; & 0 \leqslant z \leqslant r \leqslant R. \end{cases}$$

Differentiating this with respect to z we derive the condi-

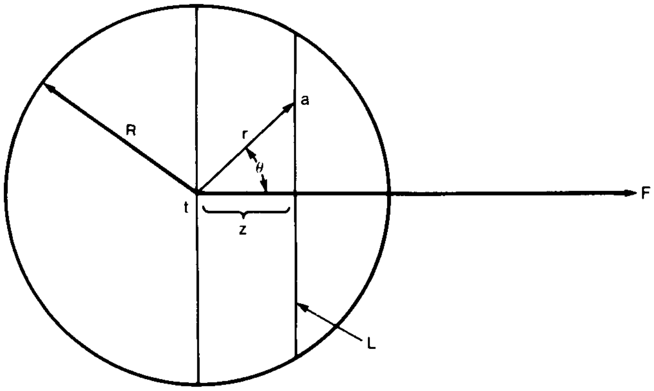

Fig. 6. Calculating the expected forward progress.

tional density

$$f(z \mid r) = \begin{cases} 0; & r < z \\ \dfrac{2}{\pi r \sqrt{1 - z^2/r^2}}; & 0 \leqslant z \leqslant r \leqslant R. \end{cases}$$

We can now calculate the expected progress given r

$$E[Z \mid r] = \frac{2}{\pi} \int_0^r \frac{z}{\sqrt{r^2 - z^2}}\, dz = \frac{2r}{\pi}.$$

We can uncondition this by using the density of (5) to obtain

$$E[Z] = \frac{2}{sR^2\pi} \left[\int_0^{R\beta^{1/2}} 2r^2 e^{-\lambda p \pi r^2/\beta}\, dr + e^{-\lambda p \pi R^2} \int_{R\beta^{1/2}}^{R} 2r^2\, dr \right]. \tag{6}$$

In [18] it is shown that

$$\int_0^x t^2 e^{-kt^2}\, dt = x \frac{e^{-kx^2}}{2k} \sum_{j=1}^{\infty} \frac{(4kx^2)^j j!}{(2j+1)!} \tag{7}$$

and thus using this in (6) and reducing we finally obtain

$$\bar{z} = \frac{2}{s\pi} e^{-Np} R \left[\frac{\beta\sqrt{\beta}}{Np} \sum_{j=1}^{\infty} \frac{(4Np)^j j!}{(2j+1)!} + \tfrac{2}{3}(1 - \beta\sqrt{\beta}) \right]. \tag{8}$$

C. Expected Throughput

We can continue along the same lines as in [12] to calculate the expected throughput for the network for each slot. For any randomly selected terminal, the expected path length between it and another randomly selected terminal is given in [16] as $d = (128/45\pi)[n/\lambda\pi]^{1/2}$. Since $\bar{z}$, as calculated in the previous section, is known, the number of hops h a randomly selected packet will take is given by $h = d/\bar{z}$. Therefore, the average number of messages delivered to their final destinations per slot, the throughput, is given by

$$\gamma(\beta, N, p) = \tfrac{45}{64} \sqrt{nN}(1-p)(1 - e^{-N/2}) p e^{-N/2} \cdot \left(\frac{\beta\sqrt{\beta}}{Np} \sum_{j=1}^{\infty} \frac{(4Np)^j j!}{(2j+1)!} + \tfrac{2}{3}(1 - \beta\sqrt{\beta}) \right).$$

The increase in the throughput γ with the square root of the number of terminals in the network is a result of the spatial reuse of the channel and was also obtained in [12]. Observe that the equation obeys our intuition for $p = 0$ or $p = 1$ where the throughput is zero, and that the $(1 - e^{-N/2})$ term is the probability that the network is connected over one hop. Once again we can show that the throughput of the system is an increasing function of β since if the function is increasing in $\alpha = \beta\sqrt{\beta}$ then it is also increasing in β. Thus, γ is linear in α with slope $m = \sum_{j=1}^{\infty} (4^j (Np)^{j-1} j!/(2j+1)!) - 2/3$ which is clearly positive since the minimum of m is 2/3. Thus as we have seen previously, increasing the capture-parameter can only increase performance.

D. Discussion of Results

In all the following graphs and tables we use normalized throughput $\gamma'(N, p, \beta) = \gamma(N, p, \beta)/\sqrt{n}$, hence eliminating the dependency of the size of the network from our equations. We must note that the square root of n dependency on the throughput is the important factor that lets us achieve, by voluntarily limiting the strength of the transmitted signal so it reaches only a subset of the nodes in the network, throughputs greater than by running the network as a one-hop ALOHA network where each node transmits with a power such that every node in the network hears the transmission. For example, in Table II for $\beta = 0.7$, we see that $\gamma' = 0.0749282$. To determine the number of terminals needed in the network to achieve throughput greater than $1/e$ we set $\sqrt{n}\gamma' > 1/e$ which implies $n \approx 24$. Thus, in a network with more than 24 terminals, it pays to voluntarily limit the transmission ranges so that on the average only $N = 4.99725$ other terminals are within hearing range. Fig. 7 demonstrates graphically the result we saw in the previous pages that increasing the capture-parameter improves system performance. Here we plot γ' as a function of p for a fixed N and various values of β. In Table II we have listed the maximum γ' over all possible N and p values for a fixed β value. We note again that the γ' is increasing in β. Observe that the spread of the optimal values of N and p over all β values is not wide. Since these values do not change substantially with β, the capture-parameter of packet radios in the network does not need to be known to a high degree of accuracy to determine the network's N and p values that achieve optimal performance. In Table II we have also listed the probability of a successful transmission as well as the expected forward progress for the same N and p values. By comparing Tables I and II, we observe that values of N and p which maximize γ' do not also maximize $P[S]$. Although it might seem intuitive that maximizing the number of successes in the network by picking an optimal transmission range R and hence by picking $N = \lambda\pi R^2$ would increase the throughput of the system, a little thought shows that this is not necessarily true. We can see this from Table I where the N values that maximize $P[S]$ are seen to be small, approximately 2.6 for $\beta = 0.7$. The network is in this case divided into many receiver–transmitter pairs in an attempt to take full advantage of the spatial reuse of the channel, and although this increases the probability of successful transmission, packets in such an environment must pass over many hops before reaching their final destinations. This tends to decrease the number of packets reaching their final destinations in any one slot, and thus reduces the throughput of the system. This tradeoff between the probability of success and the throughput of the system as governed by the number of hops between source and destination is a fundamental issue of multihop systems and occurs in several guises. For example, we have already shown that increasing β will increase γ'. This increase in γ' can result from an increased $P[S]$, an increased $\bar{z}$ (thus decreasing the average number of hops a packet takes from source to destination), or a combination of both. We see

TABLE II
OPTIMAL N AND p FOR THE FIRST MODEL FOR A GIVEN β

β	N	p	γ'	$P[E_s]$	$\bar{z}$
0.0	4.33261	0.18012	0.0584586	0.05991	0.42441
0.1	4.36181	0.18193	0.0591979	0.06285	0.40834
0.2	4.41670	0.18531	0.0605900	0.06577	0.39687
0.3	4.49194	0.18978	0.0624732	0.06879	0.38797
0.4	4.58656	0.19523	0.0648259	0.07190	0.38094
0.5	4.70160	0.20155	0.0676626	0.07526	0.37541
0.6	4.83818	0.20867	0.0710160	0.07875	0.37120
0.7	4.99725	0.21647	0.0749282	0.08242	0.36823
0.8	5.17892	0.22474	0.0794432	0.08624	0.36649
0.9	5.38063	0.23324	0.0845999	0.09022	0.36601
1.0	5.59807	0.24164	0.0904239	0.09433	0.36682

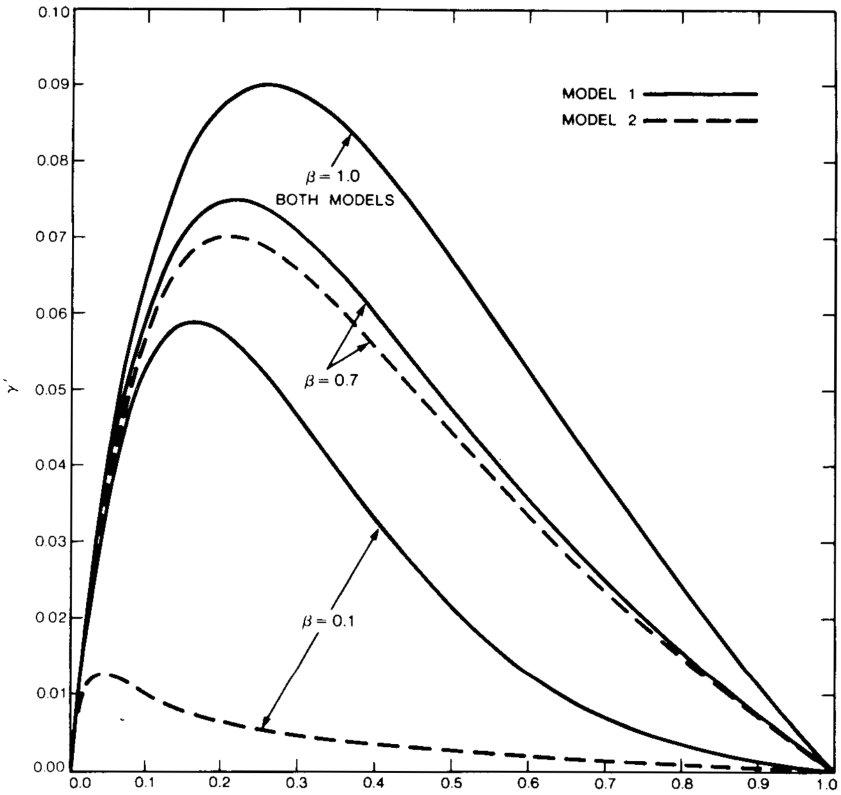

Fig. 7. γ' as a function of p for $N = 5$ and $\beta = 0.1, 0.7$, and 1.0.

in Table II that as β increases from 0 to 0.9, $P[S]$ increases and $\bar{z}$ decreases. Thus, for optimal throughput, packets must travel over more hops but they "hop" more frequently, once again showing the tradeoff between $P[S]$ and $\bar{z}$.

In Fig. 8 we show the relationship of γ' as a function of p for fixed N and β. We notice that for any N, optimal performance is degraded for small changes of p from its optimal value p^*, but that as N increases, the curves around this p^* become narrow. This variation of γ' for large N results from the fact that the transmission of any packet radio interferes with a larger number of other terminals. This increase in the number of collisions increases the sensitivity of the throughput for perturbations of the transmission probability from its optimal value.

We can unify our discussion of these results by defining the offered load per unit area to be $G = Np$. From previous results [17] for finite population slotted ALOHA networks, we known that $G = 1$ optimizes network throughput. In the multihop environment, however, connectivity of the network must be preserved. This consideration, as noted before, manifests itself in the $c = (1 - e^{-N/2})$ term appearing in the equation for γ. If p is large, $G = 1$ implies that $N = 1/p \approx 1$ which tends to disconnect the network since $c \approx 0.39$. Obviously, optimal throughput for this case would have $N > 1$ and, thus, $G > 1$. We would thus expect $G = 1$ only in cases where p is small enough to make N sufficiently large to assure connectivity. We must, however, take account of capture in discussing the offered load. In the noncapture environment we would expect the offered load that maximizes throughput to be less than that for the capture environment because the probability a transmission suffers a collision is greater for $\beta = 0$ than for $\beta = 1$. Thus, increasing $G = Np$ has a greater effect in increasing the number of expected collisions in environments with noncapture than for those with perfect capture. To check this intuition, we numerically calculated the N value that maximized γ' for fixed β and p and plotted $G = Np$ against p in Fig. 9. We see that curves for high β values dominate those for lesser values, justifying our belief that the offered load that maximizes throughput can be greater for larger β, and that as p increases so does G, illustrating the relationship of the connectivity factor c has in sparse environments. We can lend some mathematical insight into these graphs by defining the *effective number of neighbors* N' to satisfy $N = N'/(1 - e^{-N'/2})$.

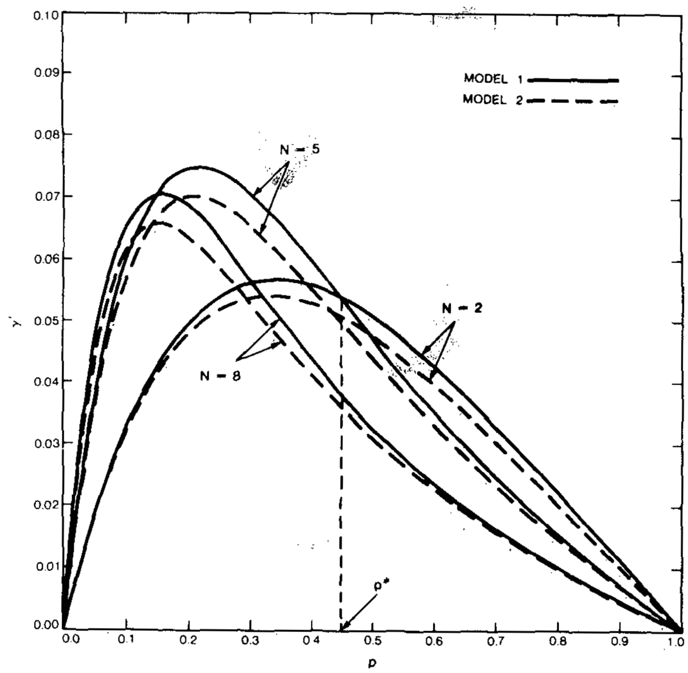

Fig. 8. γ' as a function of p for $\beta = 0.7$ and $N = 2, 5$, and 8.

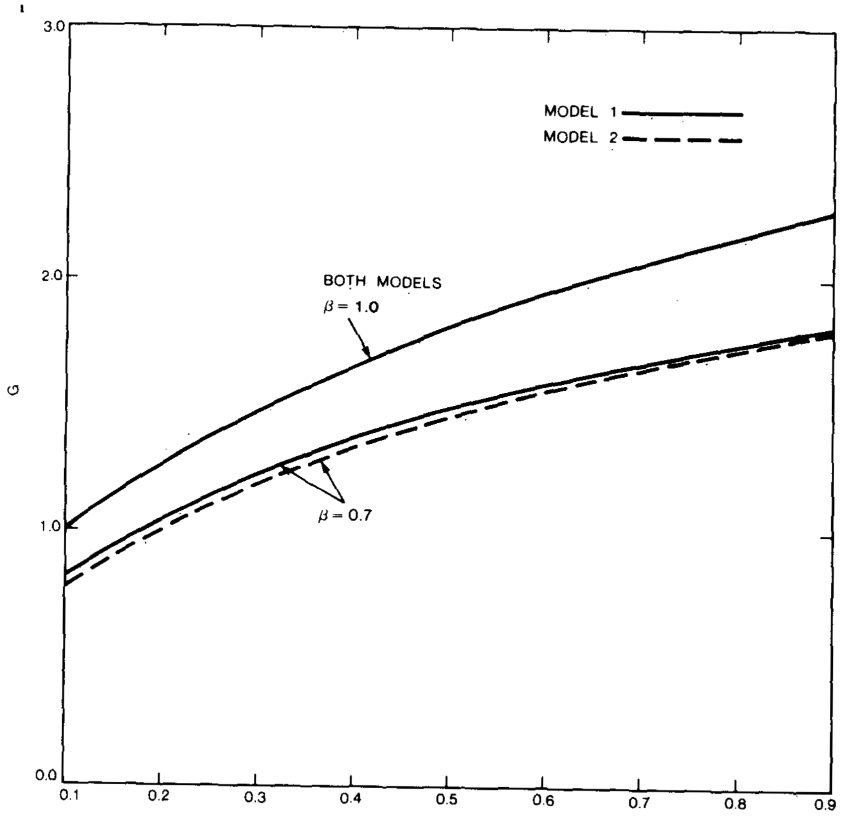

Fig. 9. $G = Np$ as a function of p for optimal N and p, and $\beta = 0.7$ and 1.0.

One then interprets N as the average number of terminals per unit area that results when we randomly distribute terminals on the plane with an average density of N' terminals per unit area, and condition upon having a connected network. The $(1 - e^{-N'/2})$ term is thus seen to be the conditional probability of hop connectivity. Using N' instead of N in calculating the offered load, $G' = pN' \approx 1$ implies that $N' \approx 1/p$ and using this in the above equation that defined N', we would have $N \approx \hat{N} = 1/p(1 - e^{-1/2p})$. To check this intuition, we used the values obtained in generating the offered load curves of Fig. 9 for $\beta = 1$ where we have seen that for low p values, $G \approx 1$. In Table III we produce the N and p values that optimize γ', as well as their product, the offered load G in the left part of the Table. In the right we tabulate the hypothesized values using the effective number of neighbors N'. We see that N and $\hat{N}$ are approximately equal and that the effective

TABLE III
THE EFFECTIVE NUMBER OF NEIGHBORS FOR OPTIMAL N AND p

N	p	$G = Np$	$\hat{N} \approx N$	N'	$G' \approx N'p$
10.07969	0.1	1.0079	10.0678	10.0121	1.0012
6.28435	0.2	1.2568	5.4471	5.9661	1.1932
4.91683	0.3	1.4750	4.1095	4.3614	1.3084
4.14327	0.4	1.6573	3.5038	3.3779	1.3511
3.62592	0.5	1.8129	3.1639	2.6733	1.3366
3.24784	0.6	1.9487	2.9477	2.1259	1.2755
2.95583	0.7	2.0690	2.7986	1.6793	1.1755
2.72158	0.8	2.1772	2.6896	1.3027	1.0421
2.52836	0.9	2.2755	2.6067	0.9774	0.8796

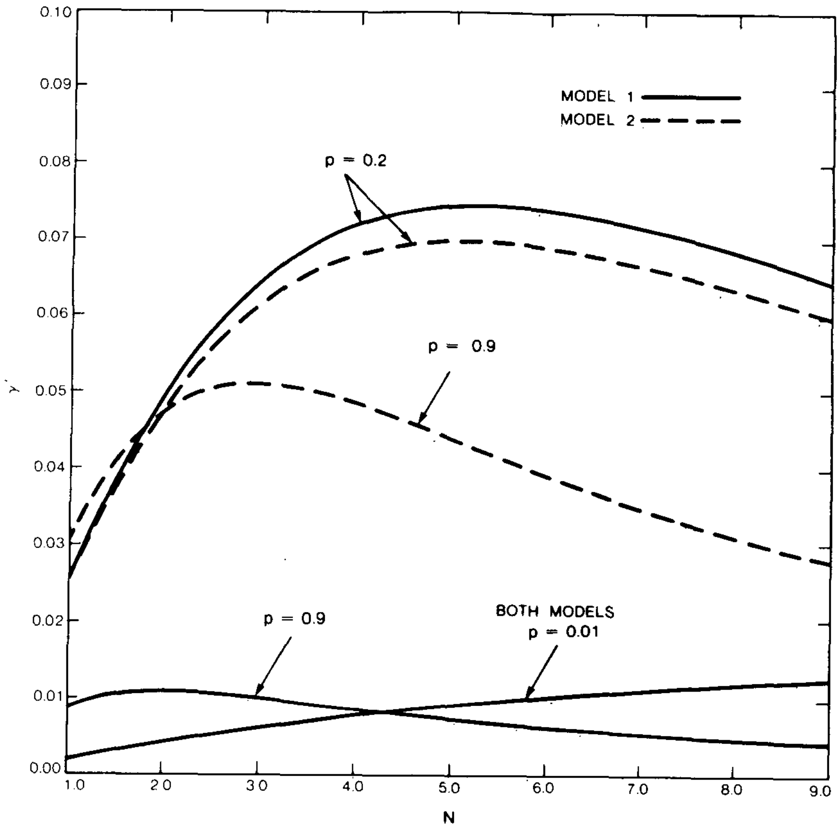

Fig. 10. γ' as a function of N for $\beta = 0.7$ and $p = 0.01, 0.2,$ and 0.9.

number of neighbors N' is strictly less that N. Comparing G and G' we see that G' is much closer to 1 throughout the range of p lending support to our previous intuitive arguments.

In our last plot for this section, Fig. 10, we graph γ as a function of N for $\beta = 0.7$ and various values of p. Observe that for the near optimal N for $p = 0.2$ (namely $N = 5$), the curve is very flat. This implies it is not necessary to determine N to a high degree of accuracy to achieve near optimal performance.

IV. ANALYSIS MODEL 2

Recall in Model 2 we assume that any other transmitter within $r\beta^{-1/2}$ of a packet radio, receiving a packet from another transmitter a distance r away, will cause a collision on the channel. We thus do not need to divide the clean area into two regions. We will be brief in describing the results of this section, since most derivations follow lines similar to those in Model 1.

A. Expected Number of Successful Receptions

The clean area for a transmitter at a distance of r from the receiver is now $r\beta^{-1/2}$, and thus the probability of this area having no other transmitters is $e^{-\lambda p \pi r^2/\beta}$. We thus have

$$P[S] = p(1-p)(1-e^{-N/2}) \int_0^R \frac{2r}{R^2} e^{-\lambda p \pi r^2/\beta} \, dr.$$

Integrating this over $[0, R]$

$$P[S] = \frac{\beta(1-p)(1-e^{-N/2})}{N} (1 - e^{-Np/\beta})$$

and hence the number of successful receptions is

$$E[\#S] = \frac{n}{N} \beta(1-p)(1-e^{-N/2})(1-e^{-Np/\beta}).$$

Once again, the expected number of successes in the network is an increasing function in β. This can best be seen by writing $E[\#S]$ as a function of β and showing that the slope is positive. Let $H(\beta) = K\beta(1 - e^{-Np/\beta})$ where $K = n(1-p)(1-e^{-N/2})/N$. Differentiating $H(\beta)$ with respect to β yields $H'(\beta) = 1 - e^{-Np/\beta}(Np/\beta + 1)$ which must be greater than zero since

$e^{Np/\beta} > 1 + Np/\beta$. It makes no sense to try to obtain the limit of the above expression for the infinite population Poisson traffic slotted ALOHA model as we did in Model 1. The reason for this concerns our definition of capture for Model 2. If we let β go to zero, the capture radius $r\beta^{-1/2}$ goes to infinity. Since all our derivations assume terminals to be Poisson distributed over the plane with parameter λ, this infinite capture radius will contain an infinity of terminals which for any $p > 0$ will have an infinite number of transmitters with probability 1, thus guaranteeing a certain collision. The equation above rightly indicates that for $\beta = 0$ the expected number of successes is zero. In Model 1 since the capture radius was limited to a maximum of R, it did make sense to let $\beta \rightarrow 0$ since the capture radius was bounded. Once again we can determine analytically that $E[\#S] < n/2$ and produce Table IV which contains the probability of success for various β values. Observe that for $\beta = 1$ these results agree with those of Model 1.

B. Expected Forward Progress

We first calculate the density for the distance between a successful transmitter and its receiver. Using the same definitions as in the previous section, we have

$$g(r) = \frac{p(1-p)(1-e^{-N/2})}{P[S]} \frac{2r}{R^2} e^{-\lambda p \pi r^2/\beta}.$$

Defining $1/s = [p(1-p)(1-e^{-N/2})]/P[S]$ we can write $g(r)$ as

$$g(r) = \frac{1}{s} \frac{2r}{R^2} e^{-\lambda p \pi r^2/\beta}.$$

We thus calculate the expected forward progress for each transmission to be

$$E[Z] = \frac{4}{s\pi R^2} \int_0^R r^2 e^{-kr^2}\, dr$$

where $k = \lambda \pi p/\beta$. Using the result from [18] to expand the integral and then simplifying, we finally obtain

$$\bar{z} = \frac{2R}{\pi(1-e^{-Np/\beta})} e^{-Np/\beta} \sum_{j=1}^{\infty} (4Np/\beta)^j \frac{j!}{(2j+1)!}. \tag{10}$$

C. Expected Throughput

Using the previous results, we can now write

$$\gamma = \tfrac{45}{64} \sqrt{n/N}\, \beta(1-p)(1-e^{-N/2}) e^{-Np/\beta} \cdot \sum_{j=1}^{\infty} (4Np/\beta)^j \frac{j!}{(2j+1)!}. \tag{11}$$

Again, this is an interesting function of β, has the square root dependency on the number of nodes in the network, and explicitly accounts for $p = 0$, $p = 1$ cases and network connectivity.

D. Discussion of Results

To compare the performance of the two models we produced the same tables and graphs for Model 2 as we did for Model 1. Earlier we observed that the two models should be identifical for $\beta = 1$ since the clean area in both models for this case are identifical. Algebraic comparison between similar formulas from both models shows that they are equal for $\beta = 1$ and the results in this section show that as β grows larger, the results of the two models are approximately equal. We therefore restrict our discussion of the curves and tables in this section to relevant differences between the two models.

TABLE IV
OPTIMAL N AND p FOR A SINGLE HOP IN THE SECOND MODEL FOR A GIVEN β

β	N	p	$P[E_s]$
0.1	1.1295	0.20379	0.02737
0.2	1.5243	0.24990	0.04467
0.3	1.8104	0.27811	0.05794
0.4	2.0412	0.29824	0.06775
0.5	2.2373	0.31373	0.07789
0.6	2.4089	0.32622	0.08578
0.7	2.5621	0.33660	0.09272
0.8	2.7011	0.34543	0.09889
0.9	2.8284	0.35307	0.10443
1.0	2.9462	0.35977	0.10946

TABLE V
OPTIMAL N AND p FOR THE SECOND MODEL FOR A GIVEN β

β	N	p	γ'	$P[E_s]$	$\bar{z}$
0.1	3.02345	0.06747	0.0136244	0.02092	0.33920
0.2	3.42712	0.10981	0.0254929	0.03610	0.34538
0.3	3.77498	0.14025	0.0360910	0.04805	0.35002
0.4	4.08648	0.16375	0.0457039	0.05787	0.35371
0.5	4.37335	0.18264	0.0545213	0.06616	0.35676
0.6	4.64158	0.19828	0.0626785	0.07330	0.35936
0.7	4.89561	0.21153	0.0702766	0.07953	0.36159
0.8	5.13830	0.22293	0.0773940	0.08503	0.36355
0.9	5.37173	0.23289	0.0840929	0.08993	0.36528
1.0	5.59807	0.24164	0.0904239	0.09433	0.36682

We observe in Table V that for Model 2 there is a wider spread for optimum N and p values for the range of β values. This is due to the larger capture radius for small β. One curious difference between this and Table II is that here as β increases so does $\bar{z}$, whereas in Model 1 we observed a decrease in the $\bar{z}$ values. The increase for Model 2 can be explained by the fact that for low values of β, small values of r, the distance between transmitter and receiver, have higher probability than greater r values because the capture radius for these values is large. As β increases, the capture radius for a fixed r decreases in Model 2 and thus larger r values are more heavily weighted and thus increase $\bar{z}$. This explains the increase in $\bar{z}$ for Model 2. In Model 1, however, the capture radius was bounded to be less than R. Thus, the clean area is small for terminals that lie close to R and this increases the probability that the projected distance will be large. This is the reason why the throughput for Model 1 is always larger than that for Model 2.

In Fig. 7 we plot γ as a function of p for $N = 5$ and various β values. Once again we see the dominance of the higher β curves. We can also observe that curves for Model 1 dominate those for Model 2. Comparison of the curves for the two models on this plot shows the dominance of throughput values of Model 1 over those of Model 2. This dominance is a result that the probability of collision is much larger in the second model for any value of N and p and hence decreases the expected throughput of the system. Fig. 9 shows again the fluctuations of the offered load for increasing p values and we note that the curve for $\beta = 0.7$ is almost identical to the same curve for Model 1. Fig. 10 shows the flatness of throughput curves for fixed β and p, over values of N.

V. CONCLUSIONS

We have analyzed two models of capture in a random planar network where slotted ALOHA was used to broadcase packets on the channel. The results of the two models are similar for capture-ratios achievable on good FM receivers and thus either could be used to analyze networks of this kind. We have seen that increasing the capture parameter increases the throughput of the network and conclude that capture is a desirable feature of the radios of such a network. The tradeoff between the probability of a successful transmission and the expected number of hops taken by a packet in the network has been delineated, and we have seen that even in ideal conditions with perfect capture and one-hop messages, no more than 21 percent of the nodes in the network, on the average, can be engaged in productive communications over any slot. The square root of n dependency on the throughput has been shown to substantially increase the throughput of the network over conventional one-hop ALOHA networks when the number of nodes in the network is sufficiently large. The critical parameter to network optimization has been shown to be the value of p, the probability of transmitting in any given slot.

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